Analyzing Lyapunov spectra of chaotic dynamical systems

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It is shown that the asymptotic spectra of finite-time Lyapunov exponents of a variety of fully chaotic dynamical systems can be understood in terms of a statistical analysis. Using random matrix theory, we derive numerical and in particular, analytical results that provide insights into the overall behavior of the Lyapunov exponents particularly for strange attractors. The corresponding distributions for the unstable periodic orbits are investigated for comparison.

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Finite-time Lyapunov exponents represent an important tool for the quantitative description of the geometrical and dynamical properties of chaotic dynamical systems [1] and apply therefore to a variety of different physical situations. Numerous recent works (see, for example, Refs. [2–6]) try to describe the spectrum of these exponents in terms of products of random matrices. The majority of these investigations focus on the maximum average Lyapunov exponent that can be estimated by taking the proper mean of an ensemble of chaotic trajectories for a sufficiently large time interval p. Details of the corresponding distribution of Lyapunov exponents (LE) are thereby not important. In order to analyze strange attractors in more detail, it is, however, crucial to know the spectrum of the finite-time LE that provides valuable information on the structures and properties emerging in phase space [1,7]. For the asymptotic (large p) Lyapunov spectra it is known [8] that, apart from a few exceptions [9], a Gaussian approximation fits the behavior around the maximum very well. Very little is known however, with respect to the overall behavior of the distribution. This is in contrast to the fact that the non-Gaussian tails of the distribution have significant influence on physical processes [10].

In the present Brief Report, we analyze finite-time Lyapunov spectra for low-dimensional discrete dynamical systems with fully developed chaos for the case of large time intervals $p \gg 1$, i.e., their asymptotic form. Our main interest is twofold. First, we explore *common* features of these spectra beyond their behavior in the vicinity of their maxima and compare them to the corresponding spectra of the unstable periodic orbits (UPOs). It turns out that these features can be understood in terms of products of random matrices. Second, we investigate the origin of spectral properties that depend on the dynamical system. System-dependent characteristics can, e.g., be due to invariant structures in phase space, such as UPOs.

It is an open question how the randomlike features of the chaotic dynamics determine the distribution of the finite-time

LE. One way to investigate this is to replace certain dynamical quantities of the original system by random ones and to study the resulting changes and/or common properties in the spectra of LE. In several recent works [11,12], such random-like modifications have been suggested. Adopting this methodology here, we will use four different ensembles of trajectories leading to different distributions for the corresponding LE. Each ensemble E_i consists of N trajectories (typically $N \approx 10^6$) of a given length $p (p \gg 1)$ and is given by (i) the distribution of the starting points of the finite trajectories and by (ii) the rule applied to generate the trajectory itself. We will then be primarily interested in the distributions of the finite-time maximum LE $\lambda = (1/p) \ln \Lambda$ belonging to these ensembles $\{E_i\}$. Λ is the largest of the absolute values of the eigenvalues belonging to the transfer matrix $M_i^{(p)} = \Pi_{k=1}^p M_i(\vec{x}_k)$ with $\{\vec{x}_k | k = 1, \ldots, p\}$ being a finite-time trajectory of the ensemble E_i. $M_i(\vec{x}_k)$ is the stability matrix belonging to the (chaotic) dynamical law $\vec{x}_{n+1} = \vec{F}(\vec{x}_n)$. The index i of M_i indicates the ensemble E_i according to which the points $\{\vec{x}_k\}$ of the trajectories are determined. It is important to note that all ensembles $\{E_i\}$ (see below) use the specific functional form of the stability matrix belonging to $\vec{F}$ but involve trajectories with different degrees of randomness. In the following, we specify the ensembles $\{E_i\}$.

E_1 consists of trajectories of length p obtained via iteration of the chaotic dynamical law $\vec{F}$. The corresponding initial conditions are distributed according to the invariant density of the map. This gives the so-called finite-time Lyapunov exponent distribution (FTLED) (see Ref. [1] and references therein). The ensemble E_2 consists of trajectories that are generated by a random variable distributed according to the invariant density of the chaotic map $\vec{F}$. This yields the bootstrap Lyapunov exponent distribution (BLED) [11,12]. Compared to the FTLED, the BLED corresponds to a dynamics with enhanced random character. The successive points of the bootstrap trajectory are completely uncorrelated. The third ensemble E_3 uses a uniformly distributed random variable for the generation of the trajectories. The range of the uniform distribution is chosen according to the phase space of the dynamical system. This case corresponds

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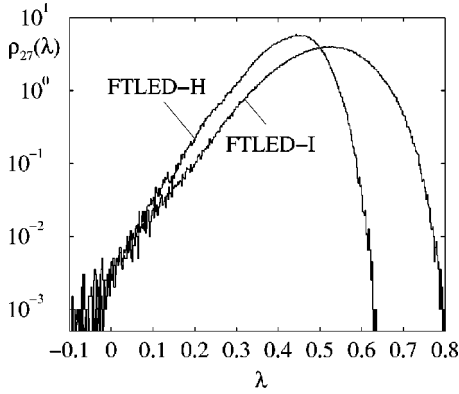FIG. 1. The FTLED (see text) of the Henon and Ikeda map for $p=27$.

FIG. 2. The BLED and RMLED (see text) of the Henon (H) and Ikeda (I) map for $p=27$.

to a random matrix simulation of the dynamical system however, respecting the form of the stability matrix **M** that belongs to the map $\vec{F}$. The resulting distribution of the Lyapunov exponents is called the random-matrix Lyapunov exponent distribution (RMLED). Within the present investigation, the ensemble E_3 possesses the highest degree of randomness. The fourth ensemble E_4 consists of the UPOs of the dynamical system $\vec{F}$ with period p and correspondingly the distribution of their maximal LE. For fixed p, the number of trajectories contained in E_4 is finite according to the topological entropy of the corresponding phase space.

Let us now explore the distributions defined above for a variety of low-dimensional fully chaotic systems. Our main goal is to analyze and understand the overall behavior of the FTLED for these systems. We begin with a simple one-dimensional (1D) example: the logistic map. Results on the FTLED for this system can be found in Ref. [9]. We note that the FTLED has a non-Gaussian form with one dominating central cusp. In comparison to this, our numerical calculations on both the BLED and RMLED show that they are smooth functions with a Gaussian-like maximum but with characteristic asymmetric tails. For the maximum Lyapunov exponent distribution of the UPOs, the exact result is a δ function, i.e., $\rho_p(\lambda)=\delta(\lambda-\ln 2)$ independent of the period p. Looking at the FTLED, one observes that the tails of the distribution are rather similar to the tails of the BLED and the RMLED while the cusp (maximum) is located exactly at $\lambda=\ln 2$. The BLED (or RMLED) reflects therefore the overall behavior of the FTLED, i.e., describes the envelope of the FTLED. The latter possesses an additional central peak at the position of the Lyapunov exponents of the UPOs. These features will in the following turn out to be common for a broad class of dynamical systems.

We focus in the following on two-dimensional systems with a strange attractor. The Henon map [13] for $a=1.4$ and $b=0.3$ and the Ikeda map [14] for $a=0.9$ and $b=6$ are prototypes of such systems. The FTLED of both are presented in Fig. 1. The ensemble E_1 consists of 10^6 trajectories of length $p=27$ for both the Henon and Ikeda map. The results are surprisingly similar to the 1D case: A characteristic envelope with asymmetric tails dominates the distribution while superimposed peaks indicate the presence of additional structures in phase space. As we shall see in the following, three regions with different functional forms of
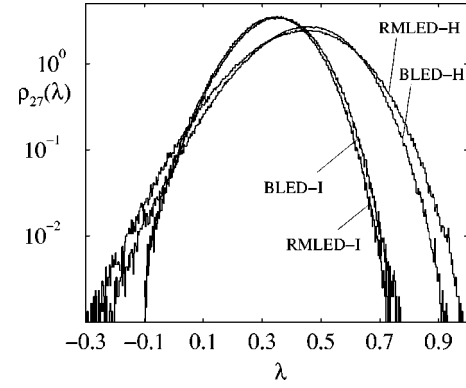
the envelope can be distinguished in the FTLED as well as the RMLED and BLED: a fast asymptotic algebraic decay for large values of λ, a Gaussian-like behavior around the maximum, and a dominating exponential decay for sufficiently small λ. Figure 2 shows the BLED and RMLED for these maps for the same length p. The position of the maximum of the RMLED is sensitive with respect to the random number intervals chosen, i.e., its location carries the information of the position of the attractor in phase space. Based on the above results and observations, we are naturally led to the following conclusion: the basic possible features of the smooth envelope of the FTLED (asymmetric structure, asymptotic tail properties) of a chaotic dynamical system are of random origin and can be obtained and understood by a corresponding study of random matrices (see below). Additional superimposed structures are signatures of, e.g., invariant sets in phase space and are therefore of exclusively deterministic dynamical origin.

To elucidate and quantify the above observations, we perform in the following an analytical investigation of the RMLED. This will allow us to thoroughly understand the behavior of the RMLED and consequently the corresponding aspects of the FTLED. We begin by introducing a fictitious dynamical system with a stability matrix $\mathbf{M}_i$ of strongly random character, i.e.,

$$\mathbf{M}_i=r_i\mathbf{A}=\begin{pmatrix} ar_i & br_i \\ cr_i & dr_i \end{pmatrix},$$

where r_i is a random variable uniformly distributed in $[0,R]$ and i labels the fictitious trajectory of length p. The simple form of the matrix $\mathbf{M}_i$ allows us to factorize the random variables $\{r_i|i=1,\ldots,p\}$ of the stability matrix $\mathbf{M}^{(p)}=\Pi_{k=1}^{p}\mathbf{M}_k$ and to reduce the problem of the product of random matrices to that of a product of random numbers. The matrix structure is then retained in the constant matrix $\mathbf{A}$ that is assumed to be nonsingular.

The distribution of the maximum LE for trajectories of length p of this system is determined as

$$\rho_p(\lambda)=\int_0^R dr_1 \int_0^R dr_2\cdots\int_0^R dr_p$$

$$\times\delta\left(\lambda-\frac{1}{p}\ln\left|\prod_{i=1}^{p}r_i\right|\Lambda_{max}\right)\prod_{l=1}^{p}\widetilde{\rho}(r_l), \quad (1)$$

where $\tilde{\rho}(z) = \Theta(z)\Theta(R-z)(1/R)$, Θ being the step function, and Λ_{max} is given by

$$\Lambda_{max} = \tfrac{1}{2}[\mathrm{Tr}\,\mathbf{A} + \mathrm{sgn}(\mathrm{Tr}\,\mathbf{A})\sqrt{(\mathrm{Tr}\,\mathbf{A})^2 - 4\det\mathbf{A}}].$$

Using the substitution $t_i = \ln(r_i|\Lambda_{max}|)$ and performing the Fourier transform of the δ function involved in Eq. (1), we obtain

$$\rho_p(\lambda) = \frac{p}{2\pi(R|\Lambda_{max}|)^p}\int_{-\infty}^{\infty} dk\, e^{-ikp\lambda}$$

$$\times \int_{-\infty}^{\ln(R|\Lambda_{max}|)} dt_1 \cdots \int_{-\infty}^{\ln(R|\Lambda_{max}|)} dt_p$$

$$\times \exp\left((1+ik)\sum_{i=1}^{p} t_i\right). \tag{2}$$

The integrations over t_i in Eq. (2) can be easily performed, leading to a pth-order pole in the complex k plane at $k=i$. This pole structure, corresponding to values of k for which the exponent of the second exponential term in Eq. (2) vanishes and for which the integration of t_i leads to singularities, is responsible for the features of the LED described above. Complex integration finally yields

$$\rho_p(\lambda) = \frac{p^p}{(p-1)!}[\ln(R|\Lambda_{max}|) - \lambda]^{p-1}$$

$$\times e^{-p[\ln(R|\Lambda_{max}|)-\lambda]}\Theta[\ln(R|\Lambda_{max}|) - \lambda]. \tag{3}$$

Equation (3) demonstrates that the exponential behavior dominates for sufficiently small values of λ. Around the maximum at $\lambda_0 = \ln(R|\Lambda_{max}|) - (1-1/p)$, the saddle point approximation ($p \gg 1$) gives us a Gaussian. For values of λ close to the maximum value $\lambda_{max} = \ln(R|\Lambda_{max}|)$, we arrive at a power-law behavior, i.e., an algebraic decay.

Although the fictitious dynamical model discussed above captures the main features of the statistical properties of the distributions of Lyapunov exponents, it is clearly desirable to investigate chaotic dynamical systems for which the RMLED can be obtained analytically. To this end, let us consider the dynamical system defined by the quadratic equations $x_{n+1} = ay_n^2 + b$; $y_n = cx_n + d$. This system possesses, for $a = c = d = 1$ and $b = -2.5$, a strange attractor that contains repeating crosses of decreasing size. For that reason we call it the cross map in the following. The average maximum Lyapunov exponent of this attractor is $\bar{\lambda} \approx 0.123$ while its fractal dimension is $d_F \approx 1.78$. The RMLED for the cross map can be calculated analytically following the line described above for our fictitious model. One peculiarity of the cross map is that one has to distinguish between the RMLED obtained through trajectories with even and odd length p. The reason is that for odd p, both eigenvalues of the stability matrix $M^{(p)}$ have the same absolute value, while for even p there are two eigenvalues different with respect to their absolute value, and one has to select the maximal one. After a tedious calculation using complex contour integration techniques, we find for the RMLED of the cross map for odd values of p the result
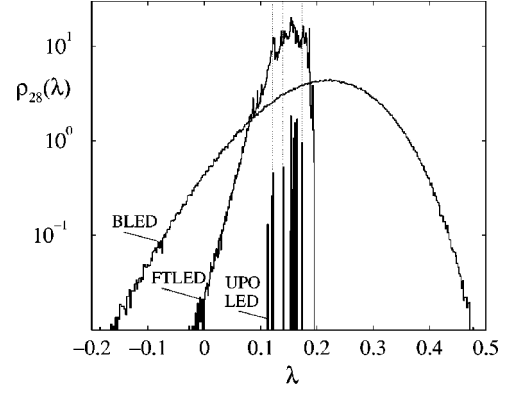

FIG. 3. The FTLED, BLED, and the distribution of the Lyapunov exponents of the unstable periodic orbits for the cross map for $p=28$.

$$\rho_p(\lambda) = \frac{2p}{p!}\left[\frac{pR_2}{R_2-R_1}\right]^p \exp[-p(\Delta-2\lambda)]$$

$$\times \sum_{j=0}^{p}\binom{p}{j}\left(\Delta - \frac{j}{p}\ln\left|\frac{R_2}{R_1}\right| - 2\lambda\right)^{p-1}$$

$$\times \Theta\left(\Delta - \frac{j}{p}\ln\left|\frac{R_2}{R_1}\right| - 2\lambda\right). \tag{4}$$

Here we have taken the random variables appearing in the stability matrix of the cross map to be uniformly distributed in the interval $[R_1, R_2]$. The parameter Δ is given as $\Delta = \ln|2acR_2|$. From Eq. (4) we see that the RMLED of the cross map is essentially a product of a single exponential and a sum of power laws. The latter possess all the same power $p-1$ and differs only with respect to the constants involved. The similarity to the RMLED result of our model system in Eq. (3) is obvious, which confirms the universality of certain features of the RMLED. The RMLED for the case of even p is given by the integral

$$\rho_p(\lambda) = 2\rho_{p/2}(\lambda)\int_{-\infty}^{\lambda} dz\, \rho_{p/2}(z) \tag{5}$$

with $\rho_{p/2}(x)$ according to Eq. (4). The integration in Eq. (5) can be performed analytically, leading to a lengthy expres-
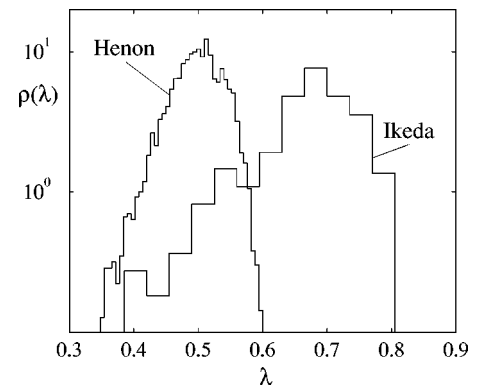

FIG. 4. The distribution of the Lyapunov exponents of the unstable periodic orbits of the Henon ($p=27$) and Ikeda map ($p=14$).

sion that will not be given here. The main characteristics of the function $\rho_p(\lambda)$ are again the features stated previously: a dominating exponential behavior for sufficiently small values of λ, a Gaussian maximum, and a fast algebraic decay for λ close to its maximum value. We have also studied the FTLED, BLED, and the distribution of the Lyapunov exponents of the UPOs for the cross map for $p=28$. The results are shown in Fig. 3. The envelope of these distributions exhibits the same features as discussed above. The additional structures present in the FTLED of the cross map are, as can be seen from Fig. 3, due to the presence of the UPOs that provide signatures of deterministic dynamical origin.

Finally, let us consider the distributions of the Lyapunov exponents of the UPOs of the Henon ($p=27$) and the Ikeda ($p=14$) map [15,16] that are presented in Fig. 4. The signs of the characteristic properties of the envelopes discussed above are also visible here. An interesting feature appears for the distribution of the LE of the UPOs of the Ikeda map: it is shifted significantly compared to the FTLED. This shift is probably due to the fact that the UPOs fail to reproduce the invariant density of the attractor, at least up to the above-considered period. It is well known that the Ikeda attractor needs a description going beyond the linear neighborhood [17].

Summarizing our results, we have demonstrated that the overall behavior of the finite-time Lyapunov exponent distributions of fully chaotic dynamical systems show general characteristics, i.e., they can be understood in terms of statistical random matrix simulations of the systems. Seemingly this holds also for the distributions of the Lyapunov exponents of the unstable periodic orbits embedded into the chaotic phase space. Since Lyapunov spectra are at the heart of our understanding of chaotic systems in general, our results apply to a variety of different physical systems.

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